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On the fixed points of the interval function $[f]([x]) = [A][x] + [b]$

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Abstract

For the interval function $[f]: \mathrm{I}(\mathbb{R}^n) \to \mathrm{I}(\mathbb{R}^n)$ defined by $[f]([x]) = [A][x] + [b]$, $|[A]|$ irreducible with $\rho(|[A]|) \geqslant 1$, we derive necessary and sufficient criteria for the existence and uniqueness of fixed points $[x]^*$. In addition, we show how $[x]^*$ can be represented by means of the input data $[A]$ and $[b]$.

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1. Introduction

When solving linear systems of equations

$$Cx = b, \quad C \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n \tag{1}$$

the Richardson splitting $C = I - A$ leads to the fixed point form

$$x = Ax + b,$$

which is equivalent to (1). Interval linear systems

$$[C]x = [b], \quad [C] \in \mathrm{I}(\mathbb{R}^{n \times n}), \quad [b] \in \mathrm{I}(\mathbb{R}^n) \tag{2}$$

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($\mathrm{I}(\mathbb{R}^{n\times n})$ = set of real $n \times n$ interval matrices, $\mathrm{I}(\mathbb{R}^n)$ = set of real interval vectors with n components) are, by definition, the collection of linear systems $Cx = b$ with $C \in [C]$, $b \in [b]$. Solving (2) means computing the solution set

$$S := \left\{x \in \mathbb{R}^n \mid Cx = b, C \in [C], b \in [b]\right\},$$

which is the union of at most 2^n intersections of finitely many half spaces (cf. [2,3,5], or [9], e.g.). Unfortunately, the determination of S is complicated. Therefore, one often confines oneself to computing an enclosure $[x]^*$ of S by an interval vector—preferably the interval hull $[x]^S \in \mathrm{I}(\mathbb{R}^n)$ which is defined as the tightest of these enclosures. An interval enclosure $[x]^*$ of S can, for instance, be obtained by using the Richardson iteration

$$[x]^{k+1} = [A][x]^k + [b], \quad k = 0, 1, \ldots, \tag{3}$$

where $[A] := I - [C]$. O. Mayer [7] showed that (3) is convergent to a unique interval vector $[x]^* \supseteq S$ if and only if the spectral radius $\rho(|[A]|)$ of the absolute value $|[A]|$ of $[A]$ (cf. Section 2) is less than one. It is easy to see that in this case $[x]^*$ is the unique solution of the interval equation

$$[x] = [A][x] + [b], \tag{4}$$

or, equivalently, $[x]^*$ is the unique fixed point of the interval function $[f]$ defined by

$$[f]([x]) := [A][x] + [b]. \tag{5}$$

Up to now the following questions on (5) are open:

(a) Does $[f]$ have a fixed point $[x]^*$ if $\rho(|[A]|) \geqslant 1$, and if so, is this fixed point unique and does it enclose S ?
(b) How does $[x]^*$ look like ?

For $|[A]|$ being irreducible we will answer these questions completely in Section 3 of our paper: We will show that $S \subseteq [x]^*$ is wrong, in general, but that for any pair $A \in [A]$, $b \in [b]$ there is at least one solution x^* of $(I - A)x = b$ which is contained in $[x]^*$. (This result even holds if $[A]$ is subject to no restriction.) We will derive necessary and sufficient conditions for $[x]^*$ to exist and to be unique. These conditions are decisively based on the observation that for irreducible matrices $|[A]|$ with $\rho(|[A]|) \geqslant 1$ either *all* components of $[x]^*$ are point intervals (as defined in Section 2) or *none*. It turns out that a fixed point $[x]^*$ can only exist if $[b]$ is a point vector. If $\rho(|[A]|) > 1$ holds, $[x]^*$ must also be a point vector, and existence and uniqueness of $[x]^*$ can be completely handled in $\mathbb{R}^n$ instead of $\mathrm{I}(\mathbb{R}^n)$. In particular, $[x]^*$ is a solution of some fixed point equation in $\mathbb{R}^n$. In the case $\rho(|[A]|) = 1$ either $[f]$ has no fixed points or infinitely many which all can then be written in the form

$$[x]^* = \check{x}^* + tv[-1, 1]. \tag{6}$$

The midpoint $\check{x}^*$ can be determined exactly and is a solution of some real linear system $x = \overset{\circ}{A}x + b$ associated with the original input data $[A]$, $[b]$. The vector v is any fixed Perron vector of $|[A]|$ and t can be any sufficiently large non-negative real number. A sharp lower bound for t can be expressed by means of $[A]$, v, and $\check{x}^*$. Therefore, together with O. Mayer's result from above, existence and uniqueness of fixed points of $[f]$ are completely clarified if $|[A]|$ is irreducible. For reducible matrices $|[A]|$ with $\rho(|[A]|) \geqslant 1$ these topics are just being studied; results will be published in a separate paper.

For irreducible matrices $|[A]|$ with $\rho(|[A]|) = 1$ the shape of $[x]^*$ is given by (6), for $\rho(|[A]|) < 1$ and particular classes of $[A]$, $[b]$ it is considered in [6].

We finally remark that the present paper has a primarily theoretical character. In the case $\rho(|[A]|) = 1$ its results can be viewed as a limit case for the practically more interesting case $\rho(|[A]|) < 1$ for which the shape of $[x]^*$ still remains open if $[A]$ is subject to no additional restrictions. The results can therefore help in finding an idea as to how this shape might look like. They can also form the starting point for further investigations of interval linear systems (2) with $[C]$ containing singular matrices $C \in [C]$.

2. Preliminaries

In addition to the notations $\mathrm{I}(\mathbb{R}^n)$, $\mathrm{I}(\mathbb{R}^{n\times n})$, $\rho(\cdot)$ in Section 1 we denote the set of real compact intervals $[a] = [\underline{a}, \overline{a}]$ by $\mathrm{I}(\mathbb{R})$. We use $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}] \in \mathrm{I}(\mathbb{R}^{n\times n})$ simultaneously without further reference, and we apply a similar notation for interval vectors. If $[a] \in \mathrm{I}(\mathbb{R})$ contains only one element a then, trivially, $\underline{a} = \overline{a} = a$. In this case we identify $[a]$ with its element writing $[a] \equiv a$ and calling $[a]$ degenerate or a point interval. Analogously, we define degenerate interval vectors/point vectors and degenerate interval matrices/point matrices, respectively. In particular, an interval vector is non-degenerate if it contains at least one entry which is not a point interval. We call $[a] \in \mathbb{R}$ symmetric if $[a] = -[a]$, i.e., if $[a] = [-a, a]$ with some real number $a \geqslant 0$. For intervals $[a], [b]$ we introduce the midpoint $\check{a} := (\underline{a} + \overline{a})/2$, the absolute value $|[a]| := \max\{|\underline{a}|, |\overline{a}|\}$, the diameter $d[a] := \overline{a} - \underline{a}$, the radius $\mathrm{rad}[a] := d[a]/2$ and the (Hausdorff) distance $q([a], [b]) := \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$. For interval vectors and interval matrices these quantities are defined entrywise, for instance, $|[A]| := (|[a]_{ij}|) \in \mathbb{R}^{n\times n}$. We assume some familiarity when working with these definitions and when applying the interval arithmetic

$$[a] \circ [b] := \left\{a \circ b \mid a \in [a], b \in [b]\right\} \in \mathrm{I}(\mathbb{R}),$$
$$[a], [b] \in \mathrm{I}(\mathbb{R}), \quad \circ \in \{+, -, \cdot, /\}, \quad 0 \notin [b] \text{ in case of '/'}.$$

Note that $[a] \circ [b]$ can be expressed by means of the bounds $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}$ of the operands $[a]$ and $[b]$. For details see, e.g., the introductory chapters of [1] or [8]. We mention here only the multiplication table (Table 1) for intervals $[a], [b] \in \mathrm{I}(\mathbb{R})$ which facilitates the understanding of some steps in the proof of Theorem 8.

Table 1
Multiplication $[a] \cdot [b]$

$[a] \cdot [b]$	$\overline{b} < 0$	$\underline{b} \leqslant 0 \leqslant \overline{b}$	$0 < \underline{b}$
$\overline{a} < 0$	$[\overline{a}\overline{b}, \underline{a}\underline{b}]$	$[\underline{a}\overline{b}, \underline{a}\underline{b}]$	$[\underline{a}\overline{b}, \overline{a}\underline{b}]$
$\underline{a} \leqslant 0 \leqslant \overline{a}$	$[\overline{a}\underline{b}, \underline{a}\underline{b}]$	$[\min\{\overline{a}\underline{b}, \underline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}]$	$[\underline{a}\overline{b}, \overline{a}\overline{b}]$
$0 < \underline{a}$	$[\overline{a}\underline{b}, \underline{a}\overline{b}]$	$[\overline{a}\underline{b}, \overline{a}\overline{b}]$	$[\underline{a}\underline{b}, \overline{a}\overline{b}]$

As usual we call $A \in \mathbb{R}^{n\times n}$ non-negative if $a_{ij} \geqslant 0$ for $i,\ j = 1, \ldots, n$, writing $A \geqslant O$ in this case. Non-negative vectors $x \geqslant 0$ and non-positive vectors $x \leqslant 0$ are defined analogously. Positive vectors x, i.e., vectors $x \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, \ldots, n$, are denoted by $x > 0$. For $A, B \in \mathbb{R}^{n\times n}$ the inequality $A \leqslant B$ means $B - A \geqslant O$.

3. Results

We first recall O. Mayer's result mentioned in Section 1 assuming here and in the sequel $[A] \in \mathrm{I}(\mathbb{R}^{n\times n})$, $[b] \in \mathrm{I}(\mathbb{R}^n)$.

Theorem 1. *For every starting vector* $[x]^0 \in \mathrm{I}(\mathbb{R}^n)$ *the iteration* (3) *converges to the same vector* $[x]^* \in \mathrm{I}(\mathbb{R}^n)$ *if and only if* $\rho(|[A]|) < 1$. *In this case* $[x]^*$ *contains the solution set*

$$S := \left\{x \in \mathbb{R}^n \mid (I - A)x = b, A \in [A], b \in [b]\right\} \tag{7}$$

and is the unique fixed point of $[f]$ *defined in* (5).

If $\rho(|[A]|) \geqslant 1$ things change. This can be seen from the following simple example.

Example 1

(a) For each fixed $r \in [-1, 1]$ the interval function $[f]([x]) := [-1, r][x]$ has the infinitely many fixed points $[x]^* = [-t, t]$, $t \geqslant 0$, and no other ones. If $-1 \leqslant r < 1$ then the solution set $S_r := \{x \in \mathbb{R} \mid (1 - a)x = 0, a \in [-1, r]\} = \{0\}$ is contained in every fixed point $[x]^*$. If $r = 1$ then $S_1 = \mathbb{R}$ since any real number solves the particular equation $x = 1 \cdot x$. Therefore no fixed point of $[f]([x]) := [-1, 1][x]$ can contain the complete solution set S_1. Note that $\rho(|[A]_r|) = 1$ for $[A]_r = ([-1, r]) \in \mathrm{I}(\mathbb{R}^{1\times 1}),\ r \in [-1, 1]$.

(b) For each fixed $r \in [-2, 2]$ the interval function $[f]([x]) = [-2, r][x]$ has the unique fixed point $[x]^* = 0$. If $-2 \leqslant r < 1$ then this fixed point contains the solution set $S_r := \{x \in \mathbb{R} \mid (1 - a)x = 0,\ a \in [-2, r]\} = \{0\}$ while $[x]^* \not\supseteq S_r = \mathbb{R}$ in the case $1 \leqslant r \leqslant 2$. Here, $\rho(|[A]_r|) = 2 > 1$ holds for $[A]_r = ([-2, r]) \in \mathrm{I}(\mathbb{R}^{1\times 1}),\ r \in [-2, 2]$.

As Example 1 shows the solution set S from (7) need not be contained in a fixed point $[x]^*$ of (5). Theorem 2 shows, however, that there is at least some connection between S and $[x]^*$.

Theorem 2. *Let* $[A] \in \mathrm{I}(\mathbb{R}^{n\times n})$, $[b] \in \mathrm{I}(\mathbb{R}^n)$, *$S$ as in* (7), *and let* $[x]^*$ *be a fixed point of* $[f]$ *defined in* (5). *Then for any linear system* $(I - A)x = b$ *with* $A \in [A]$, $b \in [b]$, *there is at least one solution which is contained in* $[x]^*$, *and* $S \subseteq [x]^*$ *holds if and only if* $I - [A]$ *is regular*, *i.e.*, *if* $I - [A]$ *does not contain any singular matrix.*

Proof. Let $x \in [x]^*$. Then $f(x) := Ax + b \in [A][x]^* + [b] = [x]^*$, and by Brouwer's fixed point theorem there is a vector $\tilde{x} \in [x]^*$ with $\tilde{x} = A\tilde{x} + b$. Since $\tilde{x} \in S$ this terminates the first part of the proof. If $S \subseteq [x]^*$ then S is bounded. Therefore, since we have already proved that any linear system $(I - A)x = b$ with $A \in [A]$ and $b \in [b]$ has at least one solution, $I - A$ must be regular for any $A \in [A]$. If, conversely, $I - [A]$ is regular then $(I - A)x = b$ is uniquely solvable for any $A \in [A]$, $b \in [b]$, and the first part of the theorem guarantees $S \subseteq [x]^*$. $\square$

We now try to generalize the results of Example 1 starting with a property of the diameter of a fixed point.

Theorem 3. *Let* $|[A]|$ *be irreducible and let* $[x]^*$ *be a fixed point of* $[f]$ *defined in* (5). *Then either* $d[x]^* = 0$ *or* $d[x]^* > 0$. *If* $\rho(|[A]|) > 1$ *then* $d[x]^* = 0$.

Proof. Assume $d[x]^*_{j_0} > 0$ for some $j_0 \in \{1, \ldots, n\}$ and choose any $i_0 \in \{1, \ldots, n\}$. Since $|[A]|$ is irreducible there is a power $|[A]|^k = (a_{ij}^{(k)})$ such that $a_{i_0 j_0}^{(k)} > 0$ [4, p. 29]. From $d[x]^* = d([A][x]^*) + d[b] \geqslant |[A]|d[x]^*$ we get $d[x]^* \geqslant |[A]|^k d[x]^*$ whence $d[x]^*_{i_0} \geqslant a_{i_0 j_0}^{(k)} d[x]^*_{j_0} > 0$. Since i_0 was arbitrary $d[x]^* > 0$ follows.

Assume now $\rho(|[A]|) > 1$ and $d[x]^* > 0$, and define the diagonal matrix $D \in \mathbb{R}^{n\times n}$ by $d_{ii} := d[x]^*_i,\ i = 1, \ldots, n$. From $d[x]^* \geqslant |[A]|d[x]^*$ we get

$$1 \geqslant \max_{1\leqslant i\leqslant n} \frac{(|[A]|d[x^*])_i}{d[x]^*_i} = \|D^{-1}|[A]|D\|_\infty \geqslant \rho(|[A]|),$$

which contradicts the assumption. $\square$

In the following theorem we study the degenerate case $d[x]^* = 0$.

Theorem 4. *Let* $|[A]|$ *be irreducible and let* $\hat{A} \in \mathbb{R}^{n\times n}$ *be the matrix which arises from* $[A]$ *by replacing the non-degenerate columns there by the corresponding columns of the identity matrix I. Define* $[f]$ *as in* (5).

(a) *The interval function* $[f]$ *has a degenerate fixed point if and only if* $[b]$ *is degenerate, i.e.*, $[b] \equiv b \in \mathbb{R}^n$, *and the linear system*

$$x = \hat{A}x + b \tag{8}$$

is solvable, *i.e.*, *b can be represented as a linear combination of the degenerate columns of* $I - [A]$. *In this case there is at least one solution* x^* *of* (8) *which satisfies*

$$x_i^* = 0 \quad \text{for all } i \in M, \tag{9}$$

where M denotes the set of indices for which the columns of $[A]$ *are non-degenerate. The degenerate fixed points* $[x]^* \equiv x^*$ *of* $[f]$ *are just the solutions of* (8) *satisfying* (9).

(b) *If* $[A]$ *has no degenerate column then* $[f]$ *has a degenerate fixed point* $[x]^* \equiv x^* \in \mathbb{R}^n$ *if and only if* $[b] \equiv 0 \in \mathbb{R}^n$. *In this case* $x^* = 0$; *it is unique in* $\mathbb{R}^n$ (*i.e.*, *there are no additional degenerate fixed points of* $[f]$).

(c) *A degenerate fixed point* $[x]^* \equiv x^* \in \mathbb{R}^n$ *of* $[f]$ *is unique in* $\mathbb{R}^n$ *if and only if either* $[A]$ *has no degenerate column—then* $b = x^* = 0$ (*cf.* (*b*))*—or the degenerate columns of* $I - [A]$ *are linearly independent.*

(d) *A degenerate fixed point* $[x]^* \equiv x^* \in \mathbb{R}$ *of* $[f]$ *is unique in* $\mathrm{I}(\mathbb{R}^n)$ (*i.e.*, *there are no additional possibly non-degenerate fixed points of* $[f]$) *if and only if one of the following conditions holds*:

(i) $\rho(|[A]|) < 1$;

(ii) $\rho(|[A]|) > 1$ *and* $[x]^* \equiv x^*$ *is unique in* $\mathbb{R}^n$ (*which is studied in* (*c*)).

In particular, $[x]^* \equiv x^*$ *is not unique in* $\mathrm{I}(\mathbb{R}^n)$ *if* $\rho(|[A]|) = 1$.

Proof

(a) Let $[x]^* \equiv x^* \in \mathbb{R}^n$ be a fixed point of $[f]$. From $d[b] \leqslant d[b] + d([A]x^*) = dx^* = 0$ we get $[b] \equiv b \in \mathbb{R}^n$ and $d([A]x^*) = 0$. This latter equality implies $x_i^* = 0$ for $i \in M$. Therefore, the fixed point equation $x^* = [A]x^* + b$ together with (9) proves the only-if-part of (a).

In order to verify the if-part of (a) choose any solution y^* of (8), replace the components y_i^* by 0 for every $i \in M$ and denote the resulting vector by x^*. Since (8) is equivalent to $(I - \hat{A})x = b$ and since by the particular form of $\hat{A}$ the ith column of $I - \hat{A}$ is zero for $i \in M$, the vector x^* remains a solution of (8) and is a fixed point of $[f]$.

(b) If $[A]$ has no degenerate column, then (9) implies $x^* = 0$, and $b = 0$ follows from (8).

(c) If $[A]$ has no degenerate column, the assertion is proved by (b). Otherwise it follows from (a) by using (8) and (9).

(d) In the case $\rho(|[A]|) < 1$ the assertion follows from Theorem 1; in the case $\rho(|[A]|) > 1$ it follows from Theorem 3. In the case $\rho(|[A]|) = 1$ no fixed point is unique in $\mathrm{I}(\mathbb{R}^n)$ as we shall see in Theorem 7. □

Remark 1. If C^+ denotes the Moore–Penrose inverse of a matrix C then it is well known that $C^+b \in \mathbb{R}^n$ is the least squares solution of $Cx = b$ of minimal Euclidean norm (cf. [10, p. 220], e.g.). Hence if $[b] \equiv b \in \mathbb{R}^n$ and if (8) is solvable, then $x^* := (I - \hat{A})^+b$ certainly is a fixed point of $[f]$ from (5): since we assumed (8) to be solvable x^* is a solution of (8); condition (9) is fulfilled since x^* has a minimal

Euclidean norm. (Modifying arbitrarily the components $y_i^*,\ i \in M$, of a solution y^* of (8) yields again a solution of (8)!)

We now consider the fixed points $[x]^*$ of $[f]$ from (5) in greater generality. To this end we start with the case $\rho(|[A]|) > 1$ which can completely be handled by Theorems 3 and 4.

Theorem 5. *Let $|[A]|$ be irreducible with $\rho(|[A]|) > 1$ and define $[f]$ by* (5).

(a) *If $d[b] \neq 0$ then $[f]$ has no fixed point.*
(b) *If $d[b] = 0$ then every fixed point $[x]^*$ of $[f]$ is degenerate*, *and existence and uniqueness of $[x]^*$ are completely handled by Theorem* 4.

Proof. Theorem 3 implies $d[x]^* = 0$. Hence every fixed point of $[f]$ is degenerate, and (a) follows from Theorem 4(a) while the remaining part of (b) is trivial. □

We now address the case $\rho(|[A]|) = 1$ which turns out to be more complicated. We first recall a result on irreducible non-negative matrices which essentially is due to Perron and Frobenius (cf. [4] or [11], e.g.).

Theorem 6. *Let $O \leqslant C \in \mathbb{R}^{n\times n}$ be irreducible. Then the following properties hold*:

(a) *The spectral radius $\rho(C)$ is positive and an algebraically simple eigenvalue of C.*
(b) *With $\rho(C)$ is associated a positive eigenvector v of C. With the exception of the positive multiples of v there are no other non-negative eigenvectors of C associated with any eigenvalue of this matrix.*
Each positive eigenvector of C is called a Perron vector of C.
(c) *The spectral radius $\rho(C)$ is a strictly increasing function of the entries of C. Moreover, if $B \in \mathbb{C}^{n\times n}$ satisfies $|B| \leqslant C$ then $\rho(B) \leqslant \rho(C)$ with equality if and only if $B = \mathrm{e}^{\mathrm{i}\phi} D^{-1} C D$, where $D = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\sigma_1}, \ldots, \mathrm{e}^{\mathrm{i}\sigma_n})$ with appropriate $\sigma_i \in \mathbb{R}$ being a complex signature matrix and $\lambda = \mathrm{e}^{\mathrm{i}\phi}\rho(C)$ being an eigenvalue of B.*

We continue with an auxiliary result.

Lemma 1. *Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$ and let $[x]^* \in \mathrm{I}(\mathbb{R}^n)$ be a fixed point of $[f]$ from* (5) *satisfying $d[x]^* > 0$. Then the following properties hold*:

(a) *The vector $[b]$ is degenerate*, *i.e.*, *$[b] \equiv b \in \mathbb{R}^n$, and $d[x]^* = |[A]|d[x]^*$, i.e., $d[x]^*$ is a Perron vector of $|[A]|$. In particular*
$$[x]^* = \check{x}^* + [-v, v] \tag{10}$$
with the Perron vector $v := d[x]^*/2 = \mathrm{rad}[x]^*$. If $\dot{A} \in [A]$ is such that $|\dot{A}| = |[A]|$ then
$$\check{x}^* = \dot{A}\check{x}^* + b \tag{11}$$

and

$$[x]^* = \dot{A}[x]^* + b = [A][x]^* + b, \tag{12}$$

in particular,

$$\dot{A}[x]^* = [A][x]^*. \tag{13}$$

For each non-degenerate symmetric entry $[a]_{i_0 j_0}$ *of* $[A]$ *we have* $\check{x}^*_{j_0} = 0$.

(b) *If* $[y]^* \neq [x]^*$ *is another fixed point of* $[f]$ *then* $q([x]^*, [y]^*) = |[A]|q([x]^*, [y]^*)$, *i.e.*, $q([x]^*, [y]^*)$ *is a Perron vector of* $|[A]|$.
The vector $[y]^*$ *can be represented as* $[y]^* = \check{y}^* + [-w, w]$ *with* $w = 0$ *or* w *being a Perron vector of* $|[A]|$, *and* $|\check{x}^* - \check{y}^*| = |[A]||\check{x}^* - \check{y}^*|$ *holds*, *i.e.*, *either* $\check{x}^* = \check{y}^*$ *or* $|\check{x}^* - \check{y}^*|$ *is again a Perron vector of* $|[A]|$.

Proof

(a) Let $d[b] \neq 0$. Then $d[x]^* = d([A][x]^*) + d[b] \geqslant |[A]|d[x]^*$ with inequality in at least one component. Hence

$$1 \geqslant \max_{1 \leqslant i \leqslant n} \frac{(|[A]|d[x]^*)_i}{d[x]^*_i} \quad \text{and} \quad 1 > \min_{1 \leqslant i \leqslant n} \frac{(|[A]|d[x]^*)_i}{d[x]^*_i}.$$

By Lemma 2.8 in [11] we get $\rho(|A|) < 1$ which contradicts our assumption. Therefore, $d[b] = 0$, $[b] \equiv b \in \mathbb{R}^n$, and the same arguments apply for showing $d[x]^* = |[A]|d[x]^*$.
The representation $[x]^* = \check{x}^* + [-v, v]$ with a Perron vector v of $|[A]|$ is a trivial consequence of what we have already proved.
Let $\dot{A} \in [A]$ be such that $|\dot{A}| = |[A]|$ holds. Then we get

$$\begin{aligned} \dot{A}\check{x}^* + [-v, v] + b &= \dot{A}\check{x}^* + |\dot{A}|[-v, v] + b = \dot{A}\check{x}^* + \dot{A}[-v, v] + b \\ &= \dot{A}(\check{x}^* + [-v, v]) + b = \dot{A}[x]^* + b \\ &\subseteq [A][x]^* + b = [x]^* = \check{x}^* + [-v, v]. \end{aligned}$$

Since the vector on the left-hand side and the vector on the right-hand side have the same diameter $2v$ we can replace '$\subseteq$' by '$=$'. This implies immediately (11), (12) and (13).
If $[a]_{i_0 j_0} = [-a_{i_0 j_0}, a_{i_0 j_0}]$ with some $a_{i_0 j_0} > 0$ then $\dot{a}_{i_0 j_0} = a_{i_0 j_0}$ or $\dot{a}_{i_0 j_0} = -a_{i_0 j_0}$. Change the sign of this entry in $\dot{A}$ and denote the resulting matrix by $\ddot{A}$. Then $\ddot{A} \in [A]$ and $|\ddot{A}| = |[A]|$, and hence $\check{x}^* = \ddot{A}\check{x}^* + b$ by (11). Subtracting this equation from (11) yields $(\dot{A} - \ddot{A})\check{x}^* = 0$. Since $\dot{a}_{ij} - \ddot{a}_{ij} = 0$ for $(i, j) \neq (i_0, j_0)$ while $\dot{a}_{i_0 j_0} - \ddot{a}_{i_0 j_0} \neq 0$ we must have $\check{x}^*_{j_0} = 0$.

(b) The representation of $[y]^*$ is either trivial or follows from (10). Define $q := q([x]^*, [y]^*) + u$ where u is any Perron vector of $|[A]|$. Then $q > 0$ and $|[A]|u = u$ leads to

$$\begin{aligned} q &= q([A][x]^* + [b], [A][y]^* + [b]) + u \\ &= q([A][x]^*, [A][y]^*) + u \\ &\leqslant |[A]|q([x]^*, [y]^*) + u \\ &= |[A]|q([x]^*, [y]^*) + |[A]|u = |[A]|q. \end{aligned} \tag{14}$$

If $q([x]^*, [y]^*) \neq |[A]|q([x]^*, [y]^*)$ then strict inequality holds in (14) at least for one component. Hence

$$1 < \max_{1 \leqslant i \leqslant n} \frac{(|[A]|q)_i}{q_i} \quad \text{and} \quad 1 \leqslant \min_{1 \leqslant i \leqslant n} \frac{(|[A]|q)_i}{q_i}.$$

This yields the contradiction $\rho(|[A]|) > 1$ by the same lemma as above.

Let $[y]^* = \check{y}^* + [-w, w]$ with $w = 0$ or w being a Perron vector of $|[A]|$. W.l.o.g. let $w \leqslant v$. (Otherwise interchange the roles of $[x]^*$ and $[y]^*$.) Then $q([x]^*, [y]^*) = |\check{x}^* - \check{y}^*| + v - w$, and from $q([x]^*, [y]^*) = |[A]|q([x]^*, [y]^*)$ we obtain $|\check{x}^* - \check{y}^*| = |[A]||\check{x}^* - \check{y}^*|$. $\square$

Remark 2

(a) Combining Theorem 4 and Lemma 1 shows that $[b]$ must be degenerate in order that $[f]$ from (5) can have a fixed point.

(b) The fixed point property (11) also follows from $d[x]^* = |[A]|d[x]^*$: This equality implies

$$d([a]_{ij}[x]_j^*) = |[a]_{ij}|d[x]_j^* = |[a]_{ij}|(\overline{x}_j^* - \underline{x}_j^*),$$

whence $[a]_{ij}[x]_j^* = \sigma|[a]_{ij}|[\underline{x}_j^*, \overline{x}_j^*]$ with $\sigma \in \{-1, 1\}$ depending on $[a]_{ij} = [-|[a]_{ij}|, \overline{a}_{ij}]$ or $[a]_{ij} = [\underline{a}_{ij}, |[a]_{ij}|]$.

(c) If $[a]_{i_0 j_0}$ is not degenerate, but does not equal $[-a_{i_0 j_0}, a_{i_0 j_0}]$ for some $a_{i_0 j_0} > 0$ then $\check{x}_{j_0}^*$ need not be zero. This can be seen from the subsequent Example 2.

(d) If there is more than one matrix $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$ there must be at least one non-degenerate symmetric entry $[a]_{ij}$ of $[A]$. Since $\check{x}_j^* = 0$ in this case, the columns of $\dot{A}$ which determine the non-zero entries of $\check{x}^*$ via (11) are the same for all choices $\dot{A} \in [A]$ such that $|\dot{A}| = |[A]|$.

Example 2. Let

$$[A] = \begin{pmatrix} \frac{1}{2} & \left[0, \frac{1}{2}\right] \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad [b] \equiv b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then $|[A]| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a Perron vector of $[A]$. The only matrix $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$ is $\dot{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Therefore,

$$\check{x}^* = \dot{A}\check{x}^* + b \;\Leftrightarrow\; \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \check{x}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\Leftrightarrow\; \check{x}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + sv = \begin{pmatrix} 1 + s \\ -1 + s \end{pmatrix}, \quad s \in \mathbb{R}.$$

Hence a solution $[x]^*$ of the fixed point equation (4) must have the form

$$[x]^* = \check{x}^* + t[-v, v] = \begin{pmatrix} [1+s-t, 1+s+t] \\ [-1+s-t, -1+s+t] \end{pmatrix}, \quad t \geqslant 0. \tag{15}$$

Since the second row of $[A]$ is degenerate, one can easily see that an interval vector of the form (15) satisfies (4) in the second component. In order to fulfill it in its first component we must have

$$\begin{aligned} &\tfrac{1}{2}[1+s-t, 1+s+t] + \left[0, \tfrac{1}{2}\right][-1+s-t, -1+s+t] \\ &\quad = [s-t, s+t]. \end{aligned} \tag{16}$$

If $-1+s+t < 0$ then the upper bound of (16) reads $\frac{1}{2}(1+s+t)+0 = s+t$ and leads to the contradiction $-1+s+t = 0$. If $-1+s-t > 0$ then the lower bound of (16) reads $\frac{1}{2}(1+s-t)+0 = s-t$ and leads to the contradiction $-1+s-t = 0$. If

$$-1+s+t \geqslant 0 \quad \text{and} \quad -1+s-t \leqslant 0 \tag{17}$$

then (16) reads

$$\tfrac{1}{2}[1+s-t, 1+s+t] + \tfrac{1}{2}[-1+s-t, -1+s+t] = [s-t, s+t],$$

i.e., (16) is fulfilled. Since (17) is equivalent to $|1-s| \leqslant t$ we end up with the following result: The interval vector $[x]^*$ is a fixed point of $[f]$ from (5) if and only if

$$[x]^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + sv + tv[-1, 1] \quad \text{with } t \geqslant |1-s|. \tag{18}$$

The vector $[x]^*$ is degenerate if and only if $t = 0$. Hence $s = 1$ and $[x]^* \equiv x^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. This confirms Theorem 4 with $\hat{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$ and $\hat{A}x^* + b = x^*$. It also confirms Lemma 1(b).

Theorem 7. *Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$ and define $[f]$ by* (5).

(a) *The interval function $[f]$ has a fixed point if and only if the following two properties hold*:
   (i) $[b] \equiv b \in \mathbb{R}^n$.
   (ii) *There is a vector $\check{x} \in \mathbb{R}^n$ such that*

$$\check{x} = \dot{A}\check{x} + b \quad \textit{for all } \dot{A} \in [A] \textit{ satisfying } |\dot{A}| = |[A]|. \tag{19}$$

(b) *If $[f]$ has a fixed point then for all sufficiently large real numbers $t > 0$ the vector*

$$[x]^* = \check{x} + tv[-1, 1] \qquad (\check{x} \textit{ from } (19),\ v \textit{ any fixed Perron vector of } |[A]|)$$

is also a fixed point of $[f]$. In particular, if $[f]$ has a fixed point then there are infinitely many ones.

Proof. (a) Let $[x]^* \in \mathrm{I}(\mathbb{R}^n)$ be some fixed point of $[f]$. Then (i) follows from Remark 2(a). In order to deduce (ii) we first assume $d[x]^* = 0$, i.e., $[x]^* \equiv x^* \in \mathbb{R}^n$. By (8) we have $x^* = \hat{A}x^* + b$ with $\hat{A}$ from Theorem 4, and by (9) we obtain $x_i^* = 0$ for all indices i which number non-degenerate columns of $[A]$. Therefore, these columns can be replaced by the corresponding ones of any $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$ without changing the result $\hat{A}x^*$. Since the remaining columns of $\hat{A}$ are degenerate and thus necessarily coincide with those of $\dot{A}$ we get $\hat{A}x^* = \dot{A}x^*$ and finally $x^* = \dot{A}x^* + b$ for all $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$. This proves (ii) with $\check{x} := x^*$. In the case $d[x]^* > 0$ the assertion (ii) follows from Lemma 1(a) with $\check{x} := \check{x}^*$.

In order to prove the converse let (i) and (ii) hold and define $\check{x}^* := \check{x}$ with $\check{x}$ from (ii). Let $v > 0$ be any Perron vector of $|[A]|$ and let $[x]^* = \check{x}^* + tv[-1, 1]$ with $t > 0$. If (13) holds for at least one matrix $\dot{A} \in [A]$ such that $|\dot{A}| = |[A]|$ then

$$\begin{aligned}[A][x]^* + b &= \dot{A}[x]^* + b = \dot{A}(\check{x}^* + tv[-1, 1]) + b \\ &= \dot{A}\check{x}^* + |\dot{A}|tv[-1, 1] + b = \check{x}^* + tv[-1, 1] = [x]^*, \end{aligned} \tag{20}$$

i.e., $[x]^*$ is a fixed point of $[f]$. We will now prove that (13) holds for all $t > 0$ sufficiently large.

Due to $v > 0$ we can choose $t \geqslant 0$ such that $\underline{x}^* \leqslant 0 \leqslant \overline{x}^*$. If $\overline{a}_{ij} \leqslant 0$ then $\dot{a}_{ij} := \underline{a}_{ij} = -|[a]_{ij}| \in [a]_{ij}$, and

$$[a]_{ij}[x]_j^* = \dot{a}_{ij}[x]_j^*. \tag{21}$$

If $\underline{a}_{ij} \geqslant 0$ then $\dot{a}_{ij} := \overline{a}_{ij} = |[a]_{ij}| \in [a]_{ij}$, and (21) holds again.

Let now $\underline{a}_{ij} < 0 < \overline{a}_{ij}$.

Case 1: $\check{a}_{ij} = 0$, i.e., $[a]_{ij} = [-a_{ij}, a_{ij}]$ with some $a_{ij} > 0$.

As at the end of the proof of Lemma 1(a) we get by (ii) $\check{x}_j^* = \check{x}_j = 0$ whence $\underline{x}_j^* = -\overline{x}_j^*$ and $[a]_{ij}[x]_j^* = |[a]_{ij}|[x]_j^* = a_{ij}[x]_j^*$. Therefore, (21) holds with $\dot{a}_{ij} := a_{ij} = |[a]_{ij}| = \overline{a}_{ij} \in [a]_{ij}$.

Case 2: $\check{a}_{ij} > 0$, i.e., $\overline{a}_{ij} = |[a]_{ij}|$.

If $\check{x}_j^* = 0$ then (21) holds with $\dot{a}_{ij} = \overline{a}_{ij}$.

If $\check{x}_j^* > 0$ then $[a]_{ij}[x]_j^* = [\min\{\underline{a}_{ij}\overline{x}_j^*, \overline{a}_{ij}\underline{x}_j^*\}, \overline{a}_{ij}\overline{x}_j^*]$. In order to fix the lower bound we remark that

$$\begin{aligned}\underline{a}_{ij}\overline{x}_j^* \geqslant \overline{a}_{ij}\underline{x}_j^* &\Leftrightarrow (\check{a}_{ij} - \mathrm{rad}[a]_{ij})(\check{x}_j^* + tv_j) \geqslant (\check{a}_{ij} + \mathrm{rad}[a]_{ij})(\check{x}_j^* - tv_j) \\ &\Leftrightarrow -\mathrm{rad}[a]_{ij}\check{x}_j^* + t\check{a}_{ij}v_j \geqslant 0 \\ &\Leftrightarrow t \geqslant \frac{\mathrm{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}_j|}{v_j}, \end{aligned} \tag{22}$$

which is true for $t \geqslant 0$ sufficiently large. Hence (21) holds with $\dot{a}_{ij} := \overline{a}_{ij} = |[a]_{ij}| \in [a]_{ij}$.

If $\check{x}_j^* < 0$ then $[a]_{ij}[x]_j^* = -([a]_{ij}(-[x]_j^*)) = -(\dot{a}_{ij}(-[x]_j^*)) = \dot{a}_{ij}[x]_j^*$ as in the case $\check{x}_j^* > 0$ provided that (22) holds.

Case 3: $\check{a}_{ij} < 0$.

Here, $[a]_{ij}[x]_j^* = -((-[a]_{ij})[x]_j^*) = -\tilde{a}_{ij}[x]_j^*$ with $\tilde{a}_{ij} := -\underline{a}_{ij} = |[a]_{ij}|$ provided that (22) is true. Setting $\dot{a}_{ij} := -\tilde{a}_{ij} = \underline{a}_{ij} = -|[a]_{ij}| \in [a]_{ij}$ results in (21).

(b) follows from the proof of (a). $\square$

In Theorem 7 we showed that there are fixed points of the form

$$[x]^* = \check{x}^* + tv[-1, 1] \tag{23}$$

provided that $[f]$ from (5) has a fixed point at all. In our next theorem we prove that *all* fixed points of $[f]$ must have this form, and we derive a sharp lower bound for t in (23) such that, in fact, $[f]([x]^*) = [x]^*$ holds. In addition, we study $\check{x}^*$ in view of uniqueness.

Theorem 8. *Let $|[A]|$ be irreducible with $\rho(|[A]|) = 1$, choose any Perron vector $v > 0$ of $|[A]|$ and define $[f]$ by* (5)*. Denote by M_{sym} the set of all indices for which the columns of $[A]$ contain at least one non-degenerate symmetric entry. Construct $[B] \in \mathbb{I}(\mathbb{R}^{n\times n})$ from $[A]$ by replacing the* jth *column of $[A]$ by the* jth *column of the identity matrix I for all $j \in M_{\text{sym}}$ and let $\overset{\circ}{A} \in [B]$ be the unique matrix which satisfies $|\overset{\circ}{A}| = |[B]|$.*

(a) *The interval function $[f]$ has a fixed point if and only if $[b]$ is degenerate, i.e., $[b] \equiv b \in \mathbb{R}^n$, and*

$$x = \overset{\circ}{A}x + b \tag{24}$$

is solvable. In this case, there is at least one solution $\check{x}$ of (24) *which satisfies*

$$\check{x}_i = 0 \quad \text{for all } i \in M_{\text{sym}}. \tag{25}$$

(b) *If $[b]$ is degenerate, i.e., $[b] \equiv b \in \mathbb{R}^n$, then for any solution $\check{x}$ of* (24) *satisfying* (25) *and for any real number t with*

$$t \geqslant m := \max\left\{ 0, \frac{\text{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}_j|}{v_j}, \frac{|\check{x}_j|}{v_j} \;\middle|\; 1 \leqslant i, j \leqslant n, \right.$$
$$\left. \check{a}_{ij} \neq 0, \text{rad}[a]_{ij} \neq 0 \right\} \tag{26}$$

the interval vector $[x]^* := \check{x} + tv[-1, 1]$ is a fixed point of $[f]$.

Conversely, if $[x]^*$ is any fixed point of $[f]$ then $[b]$ is degenerate, i.e., $[b] \equiv b \in \mathbb{R}^n$, and $[x]^*$ can be written in the form $[x]^* = \check{x}^* + tv[-1, 1]$ where $\check{x}^*$ solves (24), (25) *and t satisfies* (26) *with $\check{x} := \check{x}^*$.*

(c) *If $M_{\text{sym}} \neq \emptyset$, i.e., if there are at least two different matrices $\dot{A}$, $\ddot{A} \in [A]$ with $|\dot{A}| = |\ddot{A}| = |[A]|$, then* (24) *has at most one solution which satisfies* (25).

(d) *If $M_{\text{sym}} = \emptyset$, i.e., if there is exactly one matrix $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$, then $\dot{A} = \overset{\circ}{A}$,* (25) *is trivially true and one of the following mutually excluding cases occurs:*

(i) $\rho(\dot{A}) < 1$, *whence* (24) *has a unique solution.*

(ii) $\rho(\dot{A}) = 1$ *and* $\dot{A} \neq D^{-1}|[A]|D$ *for every signature matrix* $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ *with* $\sigma_i \in \{-1, 1\}$, *whence* (24) *has a unique solution.*

(iii) $\rho(\dot{A}) = 1$ *and* $\dot{A} = D^{-1}|[A]|D$ *for some signature matrix* $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ *with* $\sigma_i \in \{-1, 1\}$. *Here*, (24) *has no solution if and only if b cannot be represented as linear combination of the column vectors of* $I - \dot{A}$. *Otherwise it has infinitely many solutions. They are given by*

$$\check{x}^* = \check{x} + sD^{-1}v, \tag{27}$$

where $\check{x}$ *is any fixed particular solution of* (24) *and s is any real number.*

Proof. (a) Let $[f]$ have a fixed point $[x]^*$. Then the proof of Theorem 7 implies $[b] \equiv b$ and

$$\check{x}^* = \dot{A}\check{x}^* + b \tag{28}$$

for all $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$. Define M as in Theorem 4(a). Since

$$\check{x}^*_{j_0} = 0 \quad \text{for } j_0 \in M_{\mathrm{sym}} \subseteq M \tag{29}$$

(cf. Theorem 4(a) for $d[x]^* = 0$ and Lemma 1 for $d[x]^* > 0$) and since the ith columns of $\dot{A}$ and $\overset{\circ}{A}$ coincide for $i \notin M_{\mathrm{sym}}$ we get $\dot{A}\check{x}^* = \overset{\circ}{A}\check{x}^*$, and (24), (25) follow from (28), (29) with $x := \check{x} := \check{x}^*$.

Let, conversely, $[b] \equiv b$ be degenerate and let $\check{x}$ be a solution of (24) satisfying (25). Then $\dot{A}\check{x} = \overset{\circ}{A}\check{x}$ by the same arguments as above, and Theorem 7(a) implies the existence of a fixed point of $[f]$.

(b) '$\Longleftarrow$' Let $[x]^*$ be a fixed point of $[f]$. Then $[x]^*$ has the form

$$[x]^* = \check{x}^* + tv[-1, 1], \quad t \geqslant 0. \tag{30}$$

For $d[x]^* > 0$ this follows from Lemma 1 since, by virtue of Theorem 6(b), the Perron vector $d[x]^*/2$ of this lemma can be written as a positive multiple of our arbitrary Perron vector v. For $d[x]^* = 0$ the representation (30) follows at once with $t = 0$.

Now we want to prove (26) for t in (30).

If $m = 0$ then (26) holds trivially.

If $m > 0$ then by the definition of m there is some pair (i, j) such that

$$\check{x}^*_j \neq 0 \quad \text{and} \quad \check{a}_{ij} \neq 0 \quad \text{and} \quad \mathrm{rad}[a]_{ij} \neq 0 \tag{31}$$

hold together with

$$m = \frac{\mathrm{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}^*_j|}{v_j} \quad \text{or} \quad m = \frac{|\check{x}^*_j|}{v_j}.$$

If $d[x]^* = 0$ then $\mathrm{rad}[a]_{ij} \neq 0$ implies the contradiction $\check{x}^*_j = 0$ by virtue of (9). Therefore, $d[x]^* > 0$, $t > 0$, and (13) implies

$$\dot{A}[x]^* = [A][x]^* \tag{32}$$

for any $\dot{A} \in [A]$ with $|\dot{A}| = |[A]|$. Since $\dot{a}_{ij}[x]^*_j \subsetneq [a]_{ij}[x]^*_j$ would contradict (32) we get

$$\dot{a}_{ij}[x]_j^* = [a]_{ij}[x]_j^*. \tag{33}$$

From $d[x]_j^* > 0$, (31), (33) and the multiplication table in Section 2 we must have $\underline{x}_j^* \leqslant 0 \leqslant \overline{x}_j^*$, i.e., $\check{x}_j^* - tv_j \leqslant 0 \leqslant \check{x}_j^* + tv_j$ whence

$$t \geqslant \frac{|\check{x}_j^*|}{v_j}. \tag{34}$$

If $\overline{a}_{ij} < 0$ or $\underline{a}_{ij} > 0$ then $\mathrm{rad}[a]_{ij} < |\check{a}_{ij}|$, and together with (34) we obtain

$$\frac{\mathrm{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}_j^*|}{v_j} < \frac{|\check{x}_j^*|}{v_j} = m \leqslant t.$$

If $0 \in [a]_{ij}$ then $|\check{a}_{ij}| \leqslant \mathrm{rad}[a]_{ij}$ whence

$$m = \frac{\mathrm{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}_j^*|}{v_j} \geqslant \frac{|\check{x}_j^*|}{v_j}.$$

In addition, (31), (33) and the multiplication table mentioned above reveal the restrictions

$$\begin{array}{ll} \overline{a}_{ij}\underline{x}_j^* \leqslant \underline{a}_{ij}\overline{x}_j^* & \text{if } \underline{a}_{ij} \leqslant 0 < \check{a}_{ij},\ \check{x}_j^* > 0, \\ \underline{a}_{ij}\underline{x}_j^* \leqslant \overline{a}_{ij}\overline{x}_j^* & \text{if } \underline{a}_{ij} \leqslant 0 < \check{a}_{ij},\ \check{x}_j^* < 0, \\ \overline{a}_{ij}\overline{x}_j^* \leqslant \underline{a}_{ij}\underline{x}_j^* & \text{if } \check{a}_{ij} < 0 \leqslant \overline{a}_{ij},\ \check{x}_j^* > 0, \\ \underline{a}_{ij}\overline{x}_j^* \leqslant \overline{a}_{ij}\underline{x}_j^* & \text{if } \check{a}_{ij} < 0 \leqslant \overline{a}_{ij},\ \check{x}_j^* < 0. \end{array} \tag{35}$$

Expressing the bounds of the intervals $[a]_{ij}, [x]_j^*$ by means of their midpoint and radius we can rewrite the first inequality in (35) by

$$(\check{a}_{ij} + \mathrm{rad}[a]_{ij})(\check{x}_j^* - tv_j) \leqslant (\check{a}_{ij} - \mathrm{rad}[a]_{ij})(\check{x}_j^* + tv_j),$$

which is equivalent to

$$\mathrm{rad}[a]_{ij}\check{x}_j^* \leqslant \check{a}_{ij}tv_j$$

and therefore to

$$m = \frac{\mathrm{rad}[a]_{ij}}{|\check{a}_{ij}|} \cdot \frac{|\check{x}_j^*|}{v_j} \leqslant t. \tag{36}$$

The remaining three inequalities of (35) are also equivalent to (36). This proves (26).

'$\Longrightarrow$' Let now $[b] \equiv b$ be degenerate and $[x]^* = \check{x} + tv[-1, 1]$ where $\check{x}$ satisfies (24), (25) and t satisfies (26). If $t = 0$ then $m = 0$, $[x]^* \equiv \check{x}$, and the definition of m together with (25) implies $\check{x}_j = 0$ for $\mathrm{rad}[a]_{ij} \neq 0$. Hence (8) follows from (24) with $x := \check{x}$, and (9) holds too. By Theorem 4 the vector $[x]^* \equiv \check{x}$ is a fixed point of $[f]$. If $t > 0$ then $d[x]^* > 0$. We first construct a matrix $\dot{A}$ such that

$$\dot{A}[x]^* = [A][x]^*, \quad \dot{A} \in [A], \quad |\dot{A}| = |[A]| \tag{37}$$

holds which, by virtue of $\dot{A}[x]^* \subseteq [A][x]^*$, $d(\dot{A}[x]^*) = d([A][x]^*)$, is equivalent to

$$\dot{a}_{ij}[x]_j^* = [a]_{ij}[x]_j^*, \quad \dot{a}_{ij} \in [a]_{ij}, \quad |\dot{a}_{ij}| = |[a]_{ij}| \quad \text{for } i, j = 1, \ldots, n. \tag{38}$$

If (31) does not hold then $\dot{a}_{ij}$ from (38) can easily be found using (25) in the case $\check{a}_{ij} = 0$, $\mathrm{rad}[a]_{ij} > 0$. Assume now that (31) is true (which, by the way, can only happen if $m > 0$). Then $t \geqslant m \geqslant |\check{x}_j|/v_j$, i.e., $|\check{x}_j| \leqslant tv_j$, whence $0 \in [x]^*_j$. In this case $\dot{a}_{ij}$ can be constructed as in the proof of Theorem 7. (Note that by virtue of (31) not all cases must be considered there.)

From (24), (25) we get $\dot{A}\check{x} = \overset{\circ}{A}\check{x}$. Using (24) and (37) the proof terminates now as in (20).

(c) Assume that there are two solutions $\hat{x}$ and $\tilde{x}$ of (24) which satisfy (25). Then $y := \check{x} - \tilde{x}$ fulfills $y = \overset{\circ}{A}y$ and $y_i = 0$ for $i \in M_{\mathrm{sym}}$. Construct $\overset{\circ}{B}$ from $\overset{\circ}{A}$ by replacing the ith column of $\overset{\circ}{A}$ by the zero vector for every $i \in M_{\mathrm{sym}}$. Then $|\overset{\circ}{B}| \leqslant |[A]|$ with inequality in at least one entry. By virtue of Theorem 6 $\rho(\overset{\circ}{B}) < \rho(|[A]|) = 1$. Since $y_i = 0$ for $i \in M_{\mathrm{sym}}$ we obtain $y = \overset{\circ}{A}y = \overset{\circ}{B}y$, whence $y = 0$.

(d) Follows from Theorem 6 taking into account that the kernel of $I - \dot{A}$ is spanned by $D^{-1}v$ in the last case of (iii) since $\lambda = 1$ is a simple eigenvalue of $|[A]|$ and therefore of $\dot{A} = D^{-1}|[A]|D$. Note that in Theorem 6 we must set $\phi = 0$ since $\lambda = \mathrm{e}^{\mathrm{i}\phi}\rho(C)$ has to be one in our situation, and the complex signature matrix D can be chosen to be real since $B := \dot{A}$ is real. (This follows from $B = D^{-1}CD = (b_{kl}) = (\mathrm{e}^{\mathrm{i}(\sigma_l - \sigma_k)}c_{kl}) \in \mathbb{R}^{n\times n} \Leftrightarrow \mathrm{e}^{\mathrm{i}(\sigma_l-\sigma_k)} \in \mathbb{R}$ for all $k, l \in \{1, \ldots, n\}$. Choosing $k = 1$ and $l = 2, \ldots, n$ yields $\mathrm{e}^{\mathrm{i}\sigma_l} = \tau_l \mathrm{e}^{\mathrm{i}\sigma_1}$ with $\tau_l \in \{-1, 1\}$. Hence $D = \mathrm{e}^{\mathrm{i}\sigma_1}$ $\mathrm{diag}(1, \tau_2, \ldots, \tau_n)$, and σ_1 can be chosen to be zero since it has no influence on the representation of B.) $\square$

Remark 3

(a) Example 2 is an illustration of the last case in Theorem 8(d) (iii) with $D = I$. It also confirms Theorem 8(a): With the Perron vector $v = \begin{pmatrix}1\\1\end{pmatrix}$ and $\check{x}^* = \begin{pmatrix}1\\-1\end{pmatrix} + sv$ the condition (26) reduces to

$$t \geqslant \max\left\{\frac{\mathrm{rad}[a]_{12}}{|\check{a}_{12}|} \cdot \frac{|\check{x}^*_2|}{v_2}, \frac{|\check{x}^*_2|}{v_2}\right\} = |-1+s| = |1-s|$$

as required in (18).

(b) Let $\check{x}^*$ and $\check{y}^*$ be two solutions of (24) in the last case of Theorem 8(d) (iii). By (27) there are real numbers s and $\tilde{s}$ such that $\check{x}^* = \check{x} + sD^{-1}v$ and $\check{y}^* = \check{x} + \tilde{s}D^{-1}v$. Therefore, $\check{x}^* - \check{y}^* = (s - \tilde{s})D^{-1}v$, whence $|\check{x}^* - \check{y}^*| = |s - \tilde{s}|v$ as predicted at the end of Lemma 1(b).

(c) Case (ii) of Theorem 8(d) occurs, e.g., if $|[A]|$ is primitive (i.e., $\lambda = \rho(|[A]|) = 1$ is the only eigenvalue of the irreducible matrix $|[A]|$ which satisfies $|\lambda| = \rho(|[A]|)$) and if, in addition, $[A]$ has the form $[A] = [-|[A]|, O]$. In this case $\dot{A} = -|[A]|$ has no eigenvalue equal to one. Hence $\dot{A} \neq D^{-1}|[A]|D$ for every signature matrix D, and $I - \dot{A}$ is regular.

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